Power monoids and their arithmetic

Andreas Reinhart

Abstract. Let H be a unit-cancellative commutative monoid (written multiplicatively) and let $H^\times$ be its unit group. Then H is called *atomic* if every nonunit of H is a finite product of atoms of H.

For each $a \in H$, let $\mathsf{L}(a)$ be the set of all integers $n \geq 0$ such that a is a product of n atoms of H, where an *atom* is a nonunit that does not factor as a product of two nonunits; we call $\mathsf{L}(a)$ the *length set* of a (in H). Let $\mathcal{L}(H)$ be the family of all nonempty length sets in H; we call $\mathcal{L}(H)$ the *system of length sets* of H. We say that H is a BF-monoid if $\mathsf{L}(a)$ is finite and nonempty for each $a \in H$, in which case

$$\mathcal{L}(H) \subseteq \{\{0\}, \{1\}\} \cup \{L \subseteq \mathbb{N}_0 : 1 \leq |L| < \infty \text{ and } \min L \geq 2\}. \tag{1}$$

We say that H has *full system of length sets* if H is a BF-monoid and Eq. (1) holds as an equality. Furthermore, H is *fully elastic* if H is atomic and, for each rational number $r \geq 1$ smaller than

$$\rho(H) := \sup\left\{\frac{\sup \mathsf{L}(a)}{\min \mathsf{L}(a)} : a \in H \setminus H^\times\right\},$$

there is a nonunit $b \in H$ such that $r = \max \mathsf{L}(b)/\min \mathsf{L}(b)$; we call $\rho(H)$ the *elasticity* of H. Note that, if H has full system of length sets, then H is fully elastic.

It is an active research topic in factorization theory to identify monoids with full system of length sets. For example, it was shown by F. Kainrath [3] that block monoids of infinite abelian groups have full system of length sets. Furthermore, S. Frisch [2] proved that the same is true for the monoid of nonzero elements of the ring of integer-valued polynomials. Let

$$\mathcal{P}_{\text{fin},0}(\mathbb{N}_0) = \{A \subseteq \mathbb{N}_0 \mid 0 \in A \text{ and } A \text{ is finite}\},$$

and for all $A, B \in \mathcal{P}_{\text{fin},0}(\mathbb{N}_0)$, set $A+B = \{a+b \mid a \in A, b \in B\}$. Then $(\mathcal{P}_{\text{fin},0}(\mathbb{N}_0), +)$ is a unit-cancellative commutative monoid, called the reduced finitary power monoid of $\mathbb{N}_0$. There is a recent (and still open) conjecture of Fan and Tringali [1], which states that $\mathcal{P}_{\text{fin},0}(\mathbb{N}_0)$ has full system of length sets.

We offer further evidence for the validity of this conjecture. In particular, we show that $\mathcal{P}_{\text{fin},0}(\mathbb{N}_0)$ is fully elastic, and we provide a wide collection of length sets of $\mathcal{P}_{\text{fin},0}(\mathbb{N}_0)$. For instance, we show that all nonempty finite arithmetical progressions L of length $n \geq 2$ with $\min L \geq 2n$ are length sets. We also outline the limitations of our approach and discuss a potential improvement.

About the speaker. Andreas Reinhart currently holds a senior Post-Doc position at the University of Graz. He acquired his PhD in 2010 (under the supervision of Professor Franz Halter-Koch) and his habilitation in 2020 (at the University of Graz). His research interests include (but are not limited to) multiplicative ideal theory, ideal systems, factorization theory, algebraic number theory and commutative ring theory. Until now, he has published about 25 research papers in miscellaneous journals.

References

[1] Y. Fan, S. Tringali, *Power monoids: A bridge between factorization theory and arithmetic combinatorics*, J. Algebra **512** (2018), 252–294.

[2] S. Frisch, *A construction of integer-valued polynomials with prescribed sets of lengths of factorizations*, Monatsh. Math. **171** (2013), 341–350.

[3] F. Kainrath, *Factorization in Krull monoids with infinite class group*, Colloq. Math. **80** (1999), 23–30.

Institut für Mathematik und wissenschaftliches Rechnen, Karl-Franzens-Universität Graz, Heinrichstrasse 36, 8010 Graz, Austria

E-mail: andreas.reinhart@uni-graz.at · *Web*: https://imsc.uni-graz.at/reinhart/